4D Wormhole with Signature Change in the Presence of Extra Dimensions

V. Dzhunushaliev *

Institut für Mathematik, Universität Potsdam PF 601553, D-14415 Potsdam, Germany

H.-J. Schmidt †

Institut für Theoretische Physik, Freie Universität Berlin

and

Institut für Mathematik, Universität Potsdam PF 601553, D-14415 Potsdam, Germany

Abstract

A regular vacuum solution in 5D gravity on the principal bundle with the $U(1)$ structural group is proposed as a 4D wormhole. This solution has two null hypersurfaces where an interchange of the sign of some 5D metric components happens. For a 4D observer living on the base of this principal bundle this is a wormhole with two asymptotically flat Lorentzian (Euclidean) spacetimes connected by a Euclidean (Lorentzian) throat. The 4D Lorentzian observer sees these two null hypersurfaces as electric charges.

*E-Mail Addresses : dzhun@rz.uni-potsdam.de and dzhun@freenet.bishkek.su; permanent address: Dept. Theor. Phys., Kyrgyz State National University, Bishkek 720024, Kyrgyzstan

†http://www.physik.fu-berlin.de/~hjschmi hjschmi@rz.uni-potsdam.de

I. INTRODUCTION

The nice Hawking idea about a change of the signature of the spacetime metric has a problem in the classical regime: usually, a singularity appears at that point where this change takes place. The simplest explanation for this is the following: the determinant of the metric tensor $g = \det(g_{ik})$ changes its sign by changing the metric signature. Therefore at this point $g = 0$ and/or one of the scalars R or $R_{ik}R^{ik}$ or $R_{iklm}R^{iklm}$ is equal to $\pm\infty$. A detailed explanation of this fact can be found in [1] and the bibliography for this subject.

It can also be shown that the gravitational field requires additional degrees of freedom for the change of metric signature. It is easy to see if we write the metric in the vier-bein formalism:

$$ds^2 = \eta_{ab}\omega^a\omega^b, \tag{1}$$

here $\omega^a = e^a_\mu dx^\mu$; $\eta_{ab} = (+1, -1, -1, -1)$ is the Minkowski metric; e^a_μ is the vier-bein.[1] The signature of the metric is defined by η_{ab} and is not varying. It is possible that the change of the metric signature can occur as a quantum process on the spacetime foam level when η_{ab} is changed.

But below we will show that in the 5D Kaluza-Klein gravity there is a trick with interchanging of the sign between some 5D metric components that for the 4D observer is similar to the change of the signature of the 4D metric.

II. SIGNATURE CHANGE IN THE 4D WORMHOLE

In [2], [3] the following wormhole-like (WH) solution in the vacuum 5D Kaluza-Klein gravity was found:

$$ds^2_{(5)} = -\frac{r_0^2}{\Delta(r)}(d\chi - \omega(r)dt)^2 + \Delta(r)dt^2 - dr^2 - a(r)d\Omega^2, \tag{2}$$

here χ is the 5^{th} extra coordinate; r, θ, φ are the $3D$ polar coordinates; t is the time; $d\Omega^2 = d\theta^2 + \sin^2\theta d\varphi^2$ is the metric on the S^2 sphere; the subscript (5) denotes that the appropriate quantity is 5 dimensional. The equations for $\Delta(r)$ are:

---

[1] e^a_μ are the degrees of freedom of the gravitational field, this means that the gravitational equations are deduced by varying with respect to vier-bein e^a_μ.

$$\frac{\Delta''}{\Delta} - \frac{\Delta'^2}{\Delta^2} + \frac{a'\Delta'}{a\Delta} - \frac{r_0^2}{\Delta^2}\omega'^2 = 0, \tag{3}$$

$$\omega'' - 2\omega'\frac{\Delta'}{\Delta} + \omega'\frac{a'}{a} = 0, \tag{4}$$

$$\frac{\Delta'^2}{\Delta^2} + \frac{4}{a} - \frac{a'^2}{a^2} - \frac{r_0^2}{\Delta^2}\omega'^2 = 0, \tag{5}$$

$$a'' - 2 = 0. \tag{6}$$

and we see that if Δ is a solution then $-\Delta$ is also.[2]. The solution of these $5D$ Einstein's equations is

$$a = r_0^2 + r^2, \tag{7}$$

$$\Delta = \pm\frac{2r_0}{q}\frac{r^2 + r_0^2}{r^2 - r_0^2}, \tag{8}$$

$$\omega = \pm\frac{2r_0^2}{q}\frac{a'/a}{1 - \frac{2r_0^2}{a}}. \tag{9}$$

(i.e., $\omega = 2rr_0\Delta/a$), here $r_0 > 0$ and q are the same constants.

In this paper the 5D spacetime is the total space of the principal bundle with the $U(1)$ group as the structural group, the base of this bundle is the ordinary 4D spacetime [3]. This means that we condider the following part of metric (2):

$$ds^2_{(4)} = \Delta(r)dt^2 - dr^2 - a(r)d\Omega^2$$

For the metric on the total space E of the principal bundle the most natural choice of the coordinate system is the following: the y^a $(a = 5, 6, \cdots)$ coordinates[3] are chosen on the fibre (gauge group) and x^μ $(\mu = 0, 1, 2, 3)$ along the base (4D spacetime). In this case y^a are the coordinates which cover the gauge group G (the fibre of the bundle) and x^μ are the coordinates which cover the factor space E/G (the base of the bundle or 4D spacetime).

In the classical and quantum field theories (without gravitation) the strong, weak and electromagnetic interactions are characterized as a connection on the appropriate bundle with some structural group. These fields are real, therefore the corresponding total space is

---

[2]In contrast to this example, for the Schwarzschild black hole this is not the case. For the metric $ds^2 = \Delta dt^2 - dr^2/\Delta - r^2 d\Omega^2$ there we get the following equation: $\Delta' + \Delta/r - 1/r = 0$ which is not invariant under $\Delta \to -\Delta$ transformation.

[3]In our case, we restrict to the 1-dimensional fibre, i.e. $y^5 = \chi$, and the index $a = 5$.

also real in Nature. But we cannot choose the new coordinate system in which we will mix the coordinates on the fibre (gauge group) and the base (4D spacetime). This is evident because the points on the fibre are the elements of the group but the points on the base are not. In the context of this paper the metrc (2) is the metric on the total space. Once again we emphasize that the Kaluza-Klein theory in the context of this paper is the gravity on such a principal bundle, therefore we can use only the following coordinate transformations:

$$y'^a = y'^a(y^a) + f^a\left(x^\mu\right), \tag{10}$$

$$x'^\mu = x'^\mu\left(x^\mu\right). \tag{11}$$

The first term in (10) means that the choice of coordinate system on the fibre is arbitrary. The second term indicates that in addition we can move the origin of coordinate system on each fibre on the value $f^a(x^\mu)$. It is well known that such a transformation law (10) leads to a local gauge transformation for the appropriate nonabelian field (see for overview [4]). That is the (10) and (11) coordinate transformation are the most *natural* transformation for the multidimensional gravitation on the principal bundle. Of course we can use the much more generalized coordinate transformations:

$$y'^a = y'^a\left(y^a, x^\mu\right), \tag{12}$$

$$x'^\mu = x'^\mu\left(y^a, x^\mu\right). \tag{13}$$

But in this case we destroy the initial topological structure of the multidimensional spacetime, and what is even worse we mix the points of the fibre [4] with the points of the base [5]. That we do not in ordinary classical/quantum field theory (without gravitation) hence this would be a bad coordinate choice for the multidimensional gravity on the principal bundle.

The above-mentioned item is a literary description for the next exact theorem [5,6]:

Let G be the group fibre of the principal bundle. Then there is a one-to-one correspondence between the G-invariant metrics on the total space $\mathcal{X}$:

$$ds^2 = G_{AB}dx^A dx^B = g_{\mu\nu} + h(x^\mu)\left(\omega^a + A^a_\mu dx^\mu\right)^2 \tag{14}$$

---

[4] which are the elements of some group.

[5] which are the ordinary spacetimes points.

and the triples $(g_{\mu\nu}, A^a_\mu, h)$. Here G_{AB} is the multidimensional metric on the total space $(A, B = 0, 1, 2, 3, 5, 6, \cdots)$ $g_{\mu\nu}$ is Einstein's pseudo - Riemannian metric on the base; A^a_μ is the gauge field of the group G (the nondiagonal components of the multidimensional metric); $h\gamma_{ab}$ is the symmetric metric on the fibre; $\omega_a = \gamma_{ab}\omega^b$; ω^a are the one-form on the group G.

In Ref. [7] the solution (7-9) was applied for the discussion of the composite Lorentzian WH. For this goal the WH-like solution (2) (with $|r| \leq r_0$, and the sign (-) in (8), (9)) is inserted between two Reissner-Nordstöm black holes.

Below we examine two possibilities for the signs (+) and (-) in eqs. (8), (9).

A. Lorentzian wormhole with the Euclidean throat, the case of (+)

Here we examine the solution (2) with (7,8,9) in the whole region $-\infty < r < +\infty$. In this case, for the 4D spacetime (the base of the principal bundle) the following takes place:

1. By $|r| \geq r_0$ we have the ordinary 4D asymptotically flat spacetime with Lorentzian signature (from the viewpoint of the 4D observer) as $g_{tt} = \Delta > 0$.

2. By $|r| < r_0$ we have the Euclidean 4D spacetime bounded between two $ds^2_{(5)} = 0$ (located by $r = \pm r_0$) hypersurfaces as $g_{tt} = \Delta < 0$.

3. From the viewpoint of 4D the observer on the $r = \pm r_0$ hypersurfaces takes place the change of the 4D metric signature. This is a result of simple interchange the signs of the metric components G_{tt} and G_{55} on the $ds^2_{(5)} = 0$ hypersurfaces and nothing more.

Thus, the solution (2) describes the WH connecting two Lorentzian asymptotically flat regions by means of the Euclidean throat.

Of course we have a question: what happens at the hypersurfaces $r = \pm r_0$? Now we describe the properties of such hypersurface on which the interchange of the metric signature happens:

1. On this surface $ds^2_{(5)} = 0$ (χ, θ, $\varphi = const$ and $r = \pm r_0$).

2. The 5D scalar invariants $R_{(5)} = R_{(5)AB}R^{AB}_{(5)} = 0$ in the consequence of the 5D Einstein equations $R_{(5)AB} = 0$. $R_{(5)ABCD}R^{ABCD}_{(5)} \propto r_0^{-4}$ $(A = 0, 1, 2, 3, 5)$, i.e. we see that probably these two $ds^2_{(5)} = 0$ hypersurfaces do not have a singularity.

3. The 4D metric on the 4D base of the principal bundle on the 4D spacetime is (for a see eq. (7)):

$$ds^2_{(4)} = \frac{2r_0}{q} \frac{1}{1 - \frac{2r_0^2}{a}} dt^2 - dr^2 - \left(r^2 + r_0^2\right) d\Omega^2. \tag{15}$$

By $r \to \pm\infty$ we have two asymptotically flat Lorentzian spaces with the metric:

$$ds^2_{(4)} \approx \frac{2r_0}{q} dt^2 - dr^2 - r^2 d\Omega^2 \tag{16}$$

4. The 4D curvature scalar $R_{(4)}$:

$$R_{(4)} = \frac{6r_0^2}{\left(r^2 - r_0^2\right)^2}, \tag{17}$$

and it has the singularity. Thus, from the point of view of 4D the observer there is a singularity. But we emphasize once again that this singularity is not the really singularity in the consequence of the second item. This situation is similar to what happens in the Schwarzschild metric:

$$ds^2_{(4)} = \left(1 - \frac{r_g}{r}\right) dt^2 - \frac{dr^2}{1 - \frac{r_g}{r}} - r^2 d\Omega^2 \tag{18}$$

on the event horizon.

5. For the 4D observer in the Lorentzian part of this WH these two singularities look as two ($\pm$) electric charges spreaded on the $r = \pm r_0$ surfaces with the outgoing and incoming force lines of the electric field.

6. In this 5D case we cannot introduce the notion of electric charge. To see it more directly we shall look on the eq. (4), it can be rewritten in the following form:

$$\left(4\pi a \frac{\omega'}{\Delta^2}\right)' = 0. \tag{19}$$

This means that the product of the electric field $F_{01} = \omega'$ on the area $4\pi a$ of the sphere S^2 is not a conserved electric charge q. But interesting is that we can correct this field: as we see from the (19) the product of magnitude ω'/Δ^2 with the area $4\pi a$ of the sphere S^2 is the conserved flux of the corrected electric field which is proportional to the electric charge.

It is interesting to compare the metric (2) with the Reissner-Nordström solution. For this purpose we introduce the new radial coordinate $\rho = \sqrt{r^2 + r_0^2}$. Then we have our metric in the following form:

$$ds^2 = -\frac{r_0 q}{2}\left(1 - \frac{2r_0^2}{\rho^2}\right)(d\chi - \omega dt)^2 + \frac{2r_0}{q}\frac{dt^2}{1 - \frac{2r_0^2}{\rho^2}} - \frac{d\rho^2}{1 - \frac{r_0^2}{\rho^2}} - d\Omega^2. \tag{20}$$

here the area of the sphere S^2 is $4\pi\rho^2$ as for the 4D Reissner-Nordström solution but $g_{tt} = \frac{2r_0}{q}\left(1 - \frac{2r_0^2}{\rho^2}\right)^{-1}$ and $g_{\rho\rho} = \left(1 - \frac{r_0^2}{\rho^2}\right)^{-1}$ differ from the corresponding metric components of the Reissner-Nordström solution. Also the Maxwell tensor for the metric (2) is $F_{01} = \omega' = \frac{\Delta^2}{r_0}\frac{q}{\rho^2}$ whereas for the Reissner-Nordström metric we have $F_{01} = E = \frac{q}{r^2}$. Hence we can say that the metric (2) cannot be considered as the model of 4D "charge without charge". This is a simple example of a possible signature change in the presence of the extra dimensions.

B. Euclidean wormhole with the Lorentzian throat, the case (-)

Here we can precisely repeat our reasoning of subsection II A with the following interchanging: *Euclidean* $\rightleftarrows$ *Lorentzian*. Thereof we have the WH with the Lorentzian throat connecting two Euclidean asymptotically flat regions.

III. CONCLUSION

The basic idea of the Kaluza-Klein paradigm is that the extra dimensions are very small and therefore unobservable. If so then a wormhole with metric (2) can be a simple example of an Euclidean bridge between two Lorentzian regions (or the Lorentzian bridge between two Euclidean regions). We remark that this 5D construction is regular everywhere, i.e. there is not any singularity in this solution. It is remarkable that this solution is a regular *vacuum* solution for the 5D Kaluza-Klein gravity in the spirit of Einstein's idea that the right-hand side of the gravitational field equations should be <u>zero</u>.

Finally, we can say that by the assumption of the hidden extra dimensions in the Kaluza-Klein theory there is a possibility for the signature change of the 4D metric.

REFERENCES

[1] C. Hellaby, A. Sumeruk, and G. F. R. Ellis, Int. J. Mod. Phys. **D6**, 211 (1997); R. Mansouri et al., Gen. Relat. Grav. in print; G. Ellis et al. Gen. Relat. Grav. **29**, 591 (1997).

[2] V. Dzhunushaliev, Izv. Vuzov, ser. "Fizika" 78 (1993).

[3] V. Dzhunushaliev, Gen. Relat. Grav. **30**, 583 (1998).

[4] J. M. Overduin and P. S. Wesson, Phys. Rept. **283**, 303 (1997).

[5] R. Percacci, J. Math. Phys. **24**, 807 (1983).

[6] A. Salam and J. Strathdee, Ann. Phys. **141**, 316 (1982).

[7] V. Dzhunushaliev, Mod. Phys. Lett. A **13**, 2179 (1998).